Distributivity breaking and macroscopic quantum games

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Abstract

Examples of games between two partners with mixed strategies, calculated by the use of the probability amplitude as some vector in Hilbert space are given. The games are macroscopic ,no microscopic quantum agent is supposed. The reason for the use of the quantum formalism is in breaking of the distributivity property for the lattice of yes-no questions arising due to the special rules of games. The rules of the games suppose two parts: the preparation and measurement. In the first part due to use of the quantum logical orthocomplemented non-distributive lattice the partners freely choose the wave functions as descriptions of their strategies. The second part consists of classical games described by Boolean sublattices of the initial non-Boolean lattice with same strategies which were chosen in the first part. Examples of games for spin one half are given. New Nash equilibria are found for some cases. Heisenberg uncertainty relations without the Planck constant are written for the "spin one half game".

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1 Introduction

The aim of this paper is to give examples of macroscopic games between two agents, call them Alice and Bob, described by the formalism of quantum mechanics for some simple spin one half cases.

The games are macroscopic which means that differently from widely discussed now in literature [1] of the so called "quantum games" no microscopic agent like molecule, atom or photon necessarily described by quantum physics is supposed to be present.

The idea is to look for macroscopic situations in which random behavior cannot be described by the standard Kolmogorovian probability measure but by the probability amplitude represented by some vector in Hilbert space(see wide discussion of the problem in papers of A. Yu. Khrennikov and his colleagues on Växjö conferences [2]–[8].

Games look natural candidates for such situations. Our search for such situations is motivated by the discovery of Birkhoff and von Neumann [9] that the reason for the different description of chance in quantum physics is in the non-distributivity property of the lattice of properties of the microscopic system. As it is well known these lattices for simple spin one half and spin one systems are quantum logical orthocomplemented lattices.

For breaking the distributivity rule it is enough to look for situations in which disjunction for exclusive cases is defined not uniquely. The first examples of such situations for macroscopic automata were given in [10], [11]. Later in the papers [12], [13] first examples of such games imitating quantum spin one half and spin one particles with two non commuting observables measured were given.

The mathematical rule of formulating the quantum game is simple and natural. One takes some classical game with the given payoff matrix and mixed strategies (in paper [12] it was called "the foolish Alice game") and in the expression for the average profit depending on the payoff matrix and probabilities of the chosen strategy puts instead of the characteristic functions of Alice questions and Bob's answers in payoff function "operators", represented as projectors in some Hilbert space. The average profit is calculated as the mathematical expectation value of the "payoff "operator for the tensor product of wave functions representing now the strategies of Alice and Bob.

Nash equilibria are found concerning different choice of wave functions by Alice and Bob. Different representations of the lattice by projectors depend-

ing on choice of observables by Alice and Bob, parameterized for the spin case by some angle between projections of spin observables are considered. New Nash equilibria are found.

However the most interesting problem is interpretation of such games for classical situations. In some sense this is similar to the problem of "hidden variables "in quantum physics, where one tries to find some classical system, which "effectively" is described by the formalism of quantum mechanics [14].

In this paper to clarify the rules given in [12], [13] and to come closer to the "quantum casino" realization of the games one adds a new rule added to those given in the cited papers.

One must divide the game on two parts:

1. The preparation part, the rules of which are similar to those given in [12], [13].

2. Measurement of two or more non-commuting operators of the described system. This second part consists of two or more classical games, the strategies in which must be those chosen by the partners in the first part. This "must" means some following of the "tradition" chosen in the first part.

In standard quantum mechanics as it is known the frequencies of the results for measurements of different non-commuting observables with the definite prepared wave function are predetermined and cannot be arbitrary.

From the point of the axiomatic quantum theory as theory of quantum logical lattices Part 2 of our games corresponds to taking distributive sublattices of the initial non-distributive lattice with values of frequencies (or classical probabilities) prescribed by the quantum probabilistic measure (the wave function).

Preparation of wave functions of Alice and Bob means defining frequencies of definite exclusive positions of Bob's ball and Alice's questions. These frequencies however differently from classical games have less freedom in their definition. For example in our first game imitating the spin one half system with two non-commuting observables Alice and Bob due to constraint for frequencies by the wave function have free choice only defining freely one angle.

Besides the examples considered in our previous papers an example of spin one half system with three non-commuting observables of spin projections is concerned. For this case one can look for the imitation of Heisenberg uncertainty relations for spin projections in the macroscopic quantum game.

Let us describe these rules more explicitly

1. *The preparation stage.*

Alice and Bob have two quadrangles, one for Alice another for Bob. As in [12], [13]. Bob puts the ball in some vertex of his quadrangles. Alice exactly guesses to which vertex Bob put his ball. She does this by asking questions: is the ball in the vertex "a"? However Bob gives the answer "yes" not only in case the ball is in "a" but also in cases if the ball is in vertices connected by one arch with "a". It is only if the ball is in the opposite vertex that he cannot move it and definitely answers "no". This means that only "negative" answers of Bob are non-ambiguous for Alice. Alice on stage 1 fixes the number of non-ambiguous answers of Bob and calculates some frequencies for opposite vertices:

$$\omega_{1,B} = \frac{N_1}{N_1 + N_3}, \quad \omega_{3,B} = \frac{N_3}{N_1 + N_3},$$

$$\omega_{2,B} = \frac{N_2}{N_2 + N_4}, \quad \omega_{4,B} = \frac{N_4}{N_2 + N_4}$$

Now to do the game symmetric the same in supposed for Alice. Alice puts her ball to some vertex of the quadrangle and Bob must exactly guess the vertex. Only negative answers are non-ambiguous for Bob. To any graph
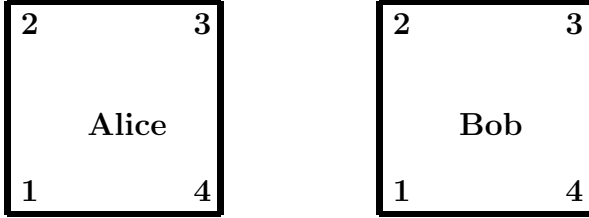

Figure 1:

of Fig. 1 corresponds non-distributive lattice Fig. 2.

Definite frequencies $\omega_{a,B}$ mean that the wave function of Bob's ball is given as ψ_B and the representation of the lattice (Fig. 2) is defined, so that two pairs of orthogonal projectors $\hat{p}_1$, $\hat{p}_3$, $\hat{p}_2$, $\hat{p}_4$ are chosen. Then $\omega_{a,B} = \langle \psi_b | \hat{p}_a | \psi_b \rangle$ as it must be for the quantum spin $\frac{1}{2}$ system with two observables – spin projections $\hat{S}_z$ and $\hat{S}_\theta$ with θ some angle being measured. Definite frequencies $\omega_{a,A}$ define the wave function of Alice's ball and some observables $\hat{S}_z$, $\hat{S}_\theta$ for Alice.
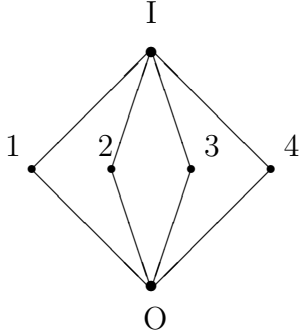
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Figure 2:

2. Measurement stage.

Two classical games are considered. Bob puts his ball only to vertices on one diagonal in the first game, let it be 1, 3. Alice asks questions trying to guess the position of Bob's ball. However, the frequencies of putting by Bob his ball to some vertices "must" be $\omega_{1,B}$, $\omega_{3,B}$ defined on the first stage. The frequencies of Alice's questions "must" be $\omega_{1,A}$, $\omega_{3,A}$. Money are paid to Alice on this stage and their amount is fixed by the payoff matrix. In the second game Bob put the ball to vertices 2, 4 with frequencies $\omega_{2,B}$, $\omega_{4,B}$ and Alice asks questions with frequencies $\omega_{2,A}$, $\omega_{4,A}$. The profits in two games are added.

The result will be given by use of the expectation value of the sum of projectors multiplied on the elements of the payoff matrix for the tensor products of two wave functions, as it was written in [12], [13]. The quantum game so formulated is the irrational game which makes it's theory different from the usual game theory. The payoff matrix is known to the players from the beginning, but as we see on the "measurement" stage it doesn't motivate their behavior. However it can motivate somehow their behavior on the first stage when the wave functions and "observables" are defined. Nash equilibria for fixed angles for observables can be understood as some "patterns" in random choice of two players.

2 Spin one half game with three observables

Here we consider more complicated game imitating particle with spin one half, for which three non-commuting observables $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$ are measured. This case is interesting because here one can imitate Heisenberg uncertainty

relations for spin projections in case of our quantum game. For simplicity we consider that same observables are measured by Alice and Bob (no difference in angles between projections is supposed). The Hasse diagram for this case is:
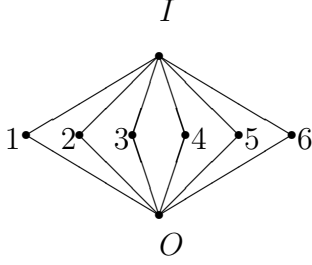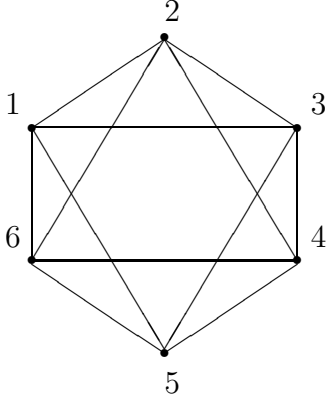

Figure 3:

Here orthogonal projectors are $1-4, 2-5, 3-6$ which correspond for the spin $\frac{1}{2}$ case to $\hat{S}_x = \pm\frac{1}{2}$, $\hat{S}_y = \pm\frac{1}{2}$, $\hat{S}_z = \pm\frac{1}{2}$.

The rule is the same, Bob can move his ball on one step depending on Alice question. For example he can move to 1 from 2, 6, 3, 5 but not from 4 etc. For Alice "no answer" on 1 means "Bob is at 4", "no answer on 2" means he is at 5 etc. Same is supposed to Alice's ball and Bob's questions. Representation of atoms of the Hasse diagram (Fig. 3) by projections is:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_3 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad A_5 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad A_6 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

The graph of the game, showing vertices to which Bob and Alice put their balls is the following.

Figure 4.

The payoff matrix is:

Table 1: The Payoff-matrix of Alice

$A\backslash B$	1	2	3	4	5	6
1	0	0	0	v_1	0	0
2	0	0	0	0	v_2	0
3	0	0	0	0	0	v_3
4	v_4	0	0	0	0	0
5	0	v_5	0	0	0	0
6	0	0	v_6	0	0	0

Then the payoff operator of the quantum game is

$$\hat{P} = v_1 A_1 \otimes B_4 + v_2 A_2 \otimes B_5 + v_3 A_3 \otimes B_6 + v_4 A_4 \otimes B_1 + v_5 A_5 \otimes B_2 + v_6 A_6 \otimes B_3$$

Here all B_i for Bob has the same form as A_i. The strategies of Alice asking questions and Bob putting the ball in their graphs (Fig.4) are described as frequencies of choices in "preparation part" given by wave functions, represented as vectors in complex Hilbert space: $\varphi_A = (\cos\alpha, e^{i\theta}\sin\alpha)$, $\psi_B = (\cos\beta, e^{i\omega}\sin\beta)$. So generally differently from real two dimensional

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space in the previous example [12], [13] one can take as in quantum spin one half physics complex space. The average profit in subsequent three "measurement" games is:

$$E_A = \langle \varphi_A | \otimes \langle \psi_B | \, \hat{P} \, | \psi_B \rangle \otimes | \varphi_A \rangle.$$

It is calculated as

$$E_A = v_1 \cos^2 \alpha \sin^2 \beta + v_2 \frac{1 + \cos \theta \sin 2\alpha}{2} \cdot \frac{1 - \cos \omega \sin 2\beta}{2} +$$

$$+ v_3 \frac{1 + \sin \theta \sin 2\alpha}{2} \cdot \frac{1 - \sin \omega \sin 2\beta}{2} + v_4 \sin^2 \alpha \cos^2 \beta +$$

$$+ v_5 \frac{1 - \cos \theta \sin 2\alpha}{2} \cdot \frac{1 + \cos \omega \sin 2\beta}{2} + v_6 \frac{1 - \sin \theta \sin 2\alpha}{2} \cdot \frac{1 + \sin \omega \sin 2\beta}{2}$$

So Nash equilibria can be found by analyzing the function $E_A(\alpha, \beta, \theta, \omega)$. The simplest case is when $\theta = \omega = 0$ and φ_A, ψ_B are real. For this case define

$$a = v_1, \; b = v_4, \; c = -\frac{v_2 + v_3 + v_5 + v_6}{4}, \; d = \frac{v_2 + v_3 - v_5 - v_6}{4}$$

then $E_A = H(\alpha, \beta)$, where

$$H(\alpha, \beta) = a \cos^2 \alpha \sin^2 \beta + b \sin^2 \alpha \cos^2 \beta + c(1 - \sin 2\alpha \sin 2\beta) + d(\sin 2\alpha - \sin 2\beta)$$

To find Nash equilibria one must look for intersection points of curves of reaction of Bob and Alice [12]. Three cases were investigated by us.

- a=7, b=1, c=-2, d=1,5. Nash equilibrium exists for $\alpha = \beta = \frac{\pi}{8}$. The value of the payoff at this point is equal to 2.

- a=1, b=1, c=-2, d=0. No Nash equilibrium exists for this case.

- a=1, b=10, c=-6, d=4. Nash equilibrium exists for $\alpha = 87,9^0$, $\beta = 69,2^0$, $E_A = 4,6$.

3 Heisenberg's uncertainty relations

As we said before, the game consists of two parts:

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1. "preparation"when the non-distributive quantum logical lattice was used, leading to the choice of Alice and Bob of their wave functions.

2. measurement, described by three different games, using orthogonal vertices of the graph (fig.3) and described by frequencies obtained from part 1.

As it is well known from quantum mechanics there are Heisenberg's uncertainty relations for spin projections, so that if

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z,$$

then for dispersions one has

$$D_\psi S_x \cdot D_\psi S_y \geqslant \frac{\hbar^2}{4}(E_\psi S_z)^2 \tag{1}$$

As it was shown in [10], [11] earlier these relations for graph are equivalent to relation for frequencies obtained from the wave function:

$$p_1 p_4 p_2 p_5 \geq \frac{1}{16}(p_3 - p_6)^2 \tag{2}$$

Here p_i-frequencies for Alice. Same relation is valid for Bob. In our case

$$p_1 = \cos^2\alpha, \quad p_2 = \frac{1 + \cos\theta \sin 2\alpha}{2}, \quad p_3 = \frac{1 + \sin\theta \sin 2\alpha}{2},$$

$$p_4 = \sin^2\alpha, \quad p_5 = \frac{1 - \cos\theta \sin 2\alpha}{2}, \quad p_6 = \frac{1 - \sin\theta \sin 2\alpha}{2} \tag{3}$$

Then (2) means $\sin^2 2\alpha \leq 1$ which is always valid.

In our case of three classical games with probabilities prescribed by (3)one can consider measuring three random variables A_1, A_2, A_3 taking values ± 1 and calculate dispersions and expectation values

$$D(A_1) = \sin^2 2\alpha, \qquad D(A_2) = 1 - (\cos\theta \sin 2\alpha)^2, \qquad E(A_3) = \sin\theta \sin 2\alpha$$

So one obtains

$$D(A_1)D(A_2) \geq (E(A_3))^2 \tag{4}$$

equivalent to (2). Here differently from(1) we put $\hbar = 1$ and there is no $\frac{1}{2}$ as it was for spin variable. However if one includes in the notion of observable the "payment" defined by the payoff matrix, then for all equal v in the payoff matrix one can see that dimensional "price" can play the role of the Planckean constant.

4 Interference terms

It is easy to see that if one looks on probability (frequency) terms in different "measurement" classical games there are typical quantum interference terms. For example for the game in introduction one has in the first "measurement game" $p_1 = \cos^2 \alpha$, $p_3 = \sin^2 \alpha$ but in the second "measurement game" there are $p_2 = \cos^2(\alpha - \theta_a)$, $p_4 = \sin^2(\alpha - \theta_a)$ with fixed θ_a [12]. So

$$p_2 = (\cos \alpha \cos \theta_a - \sin \alpha \sin \theta_a)^2 = \cos^2 \theta_a p_1 + \sin^2 \theta_a p_3 - \sin 2\theta_b \sqrt{p_1} \sqrt{p_3}.$$

Same can be seen for p_4.

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